# ON IDENTITIES GENERATED BY COMPOSITIONS OF POSITIVE INTEGERS

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Abstract. We prove astonishing identities generated by compositions of positive integers. In passing, we obtain two new identities for Stirling numbers of the first kind. In the two last sections we clarify an algebraic sense of these identities and obtain several other structural close identities.

## 1. Introduction

Recall (cf.[2]) that a composition of a positive integer $n$ is a way of writing $n$ as a sum of a sequence of positive integers. These integers are called parts of a composition. Thus to a composition of $n$ with $r$ parts corresponds $r$-fold vector $(k_1, ..., k_r)$ of positive integer components with the condition $k_1 + k_2 + ... + k_r = n$. From the definition it follows that, in contrast to partitions of $n$, the order of parts matters. Note that the set of all solutions of the Diophantine equation $k_1 + k_2 + ... + k_r = n,\ k_i \geq 1$ is the set of all compositions with $r$ parts. We start with two examples.

**Example 1.** Let $k = 3$. We have the following compositions of $3 : 1+1+1 = 1 + 2 = 2 + 1 = 3$.

Let us map a composition $k = k_1 + k_2 + ... + k_r$ to the following product of binomial coefficients: $\binom{n}{k_1}\binom{n}{k_2} \cdot ... \cdot \binom{n}{k_r}$ and all compositions of $k$ we map to the sum of such products, where the summand are taken with the sign $(-1)^{k-r}$. After summing the products with the same sets of factors, we obtain a liner combinations of such products. In our case $k = 3$, we have the following linear combination of products of binomial coefficients:

$$c_3(n) = \binom{n}{1}^3 - 2\binom{n}{1}\binom{n}{2} + \binom{n}{3}. \tag{1}$$

It is easy to verify that

$$c_3(n) = \binom{n+2}{3}. \tag{2}$$

**Example 2.** We have the following compositions of $k = 4 : 1 + 1 + 1 + 1 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 3 = 3 + 1 = 2 + 2 = 4$.

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Thus we have the following linear combination of products of binomial coefficients:

$$c_4(n) = \binom{n}{1}^4 - 3\binom{n}{1}^2\binom{n}{2} + 2\binom{n}{1}\binom{n}{3} + \binom{n}{2}^2 - \binom{n}{4} \tag{3}$$

and it is easy to verify that

$$c_4(n) = \binom{n+3}{4}. \tag{4}$$

In general, we obtain the following.

**Theorem 3.**

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+\ldots+k_r=k,\ k_i\geq 1}\ \prod_{i=1}^{r}\binom{n}{k_i} = \binom{n+k-1}{k}. \tag{5}$$

In cases $k=3$ and $k=4$, formula (5), evidently, leads to Examples 1-2. It is interesting to note (using a simple induction) that the $k$-th polynomial in $n$ of the sequence $\{\binom{n+k-1}{k}\}$ is the partial sum of values of the $(k-1)$-th one:

$$\sum_{j=1}^{n}\binom{j+k-2}{k-1} = \binom{n+k-1}{k}. \tag{6}$$

## 2. An equivalent form of identity (5)

We calculate the interior sum in (5) in a combinatorial way. First, let us consider also zero parts in the compositions of $k$. In this case we have the sum

$$\Sigma_1 = \sum_{k_1+k_2+\ldots+k_r=k,\ k_i\geq 0}\ \prod_{i=1}^{r}\binom{n}{k_i}. \tag{7}$$

To calculate this sum, suppose that we have $rn$ white points and mark $k$ from them. This we can do in $\binom{rn}{k}$ ways. On the other hand, we can mark $k_1$ from $n$ points (since the white points are indistinguishable, we can choose any $n$ points), $k_2$ from another $n$ points, etc. Thus we immediately obtain the equality

$$\Sigma_1 = \binom{rn}{k}. \tag{8}$$

To calculate the required interior sum in (5)

$$\Sigma_2 = \sum_{k_1+k_2+\ldots+k_r=k,\ k_i\geq 1}\ \prod_{i=1}^{r}\binom{n}{k_i}, \tag{9}$$

we should remove zero parts in $\Sigma_1$ (7), using "include-exclude" formula. Hence, we find

$$\Sigma_2 = \binom{rn}{k} - \binom{r}{1}\binom{(r-1)n}{k} +$$

$$\binom{r}{2}\binom{(r-2)n}{k} - ... + (-1)^{r-1}\binom{r}{r-1}\binom{n}{k}. \tag{10}$$

Now, by (9)-(10), we see that (5) is equivalent to the identity

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{j=0}^{r-1}(-1)^j\binom{r}{j}\binom{n(r-j)}{k} = \binom{n+k-1}{k},$$

or, putting $i = r - j$, to the identity

$$\sum_{r=1}^{k}\sum_{i=1}^{r}(-1)^i\binom{r}{i}\binom{ni}{k} = (-1)^k\binom{n+k-1}{k}. \tag{11}$$

Changing here the order of summing, we have

$$\sum_{i=1}^{k}\sum_{r=i}^{k}(-1)^i\binom{r}{i}\binom{ni}{k} =$$

$$\sum_{i=1}^{k}(-1)^i\binom{ni}{k}\sum_{r=i}^{k}\binom{r}{i} = (-1)^k\binom{n+k-1}{k}. \tag{12}$$

As is well known,

$$\sum_{r=i}^{k}\binom{r}{i} = \binom{k+1}{i+1}.$$

Therefore, (5) is equivalent to the identity:

$$\sum_{i=1}^{k}(-1)^i\binom{ni}{k}\binom{k+1}{i+1} = (-1)^k\binom{n+k-1}{k}. \tag{13}$$

## 3. (13) as a polynomial identity in $n$

Unfortunately, we are not able to give a direct inductive proof of (13). Note that (13) means the equality between two polynomials in $n$ of degree $k$. Therefore, for a justification of (13), it is natural to use Stirling numbers of the first kind with the generating polynomial for them ([1]):

$$x(x-1)\cdot ... \cdot(x-n+1) = \sum_{j=1}^{n} s(n,j)x^j,\ n \geq 1. \tag{14}$$

Writing (13) in the form

$$\sum_{i=1}^{k}(-1)^i in(in-1)\cdot ... \cdot(in-k+1)\binom{k+1}{i+1} =$$

$$(-1)^k(n+k-1)(n+k-2)\cdot\ldots\cdot n, \tag{15}$$

by (14), we have

$$\sum_{i=1}^{k}(-1)^i\binom{k+1}{i+1}\sum_{t=1}^{k}s(k,t)(in)^t =$$

$$(-1)^k\sum_{t=1}^{k}s(k,t)(n+k-1)^t. \tag{16}$$

In the left hand side of (16), the coefficient of $n^t$ equals

$$s(k,t)\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}i^t =$$

$$-s(k,t)\sum_{j=1}^{k+1}(-1)^j\binom{k+1}{j}(j-1)^t =$$

$$-s(k,t)\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}(j-1)^t + s(k,t)(-1)^t.$$

Since $t \le k$, then the $(k+1)$-th difference

$$\Delta^{k+1}[(j-1)^t] = \sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}(j-1)^t = 0$$

and we conclude that for $t \ge 1$

$$Coef_{n^t}\Big(\sum_{i=1}^{k}(-1)^i in(in-1)\cdot\ldots\cdot(in-k+1)\binom{k+1}{i+1}\Big) = (-1)^t s(k,t). \tag{17}$$

In the right hand side of (16), the coefficient of $n^t$ equals

$$(-1)^k\sum_{j=0}^{k}s(k,j)Coef_{n^t}(n+k-1)^j =$$

$$(-1)^k\sum_{j=t}^{k}s(k,j)\binom{j}{t}(k-1)^{j-t}.$$

Thus, comparing with (17), we conclude that identity (13) is equivalent to the identity

$$\sum_{j=t}^{k}\binom{j}{t}s(k,j)(k-1)^{j-t} = (-1)^{k+t}s(k,t). \tag{18}$$

Further we need two lemmas.

## 4. Lemmas

**Lemma 4.** *For $1 \le t \le k$, we have*

$$\sum_{j=t+1}^{k} \binom{j}{t} s(k,j) = ks(k-1,t). \tag{19}$$

*Proof.* We prove the lemma in the form:

$$\sum_{i=1}^{k-t} \binom{t+i}{t} s(k,t+i) = ks(k-1,t),\ 1 \le t \le k. \tag{20}$$

We use induction over $k$. Note that (20) is valid for $k=1$ and $t \ge 1$. Suppose that

$$\sum_{i=1}^{k-1-t} \binom{t+i}{t} s(k-1,t+i) = (k-1)s(k-2,t),\ t \ge 1, \tag{21}$$

or, the same, changing the summing index $i := i-1$,

$$\sum_{i=2}^{k-t} \binom{t+i-1}{t} s(k-1,t+i-1) = (k-1)s(k-2,t),\ t \ge 1, \tag{22}$$

or

$$\sum_{i=1}^{k-t} \binom{t+i-1}{t} s(k-1,t+i-1) =$$

$$(k-1)s(k-2,t) + s(k-1,t),\ t \ge 1. \tag{23}$$

For $t \ge 2$, put in (21) $t := t-1$. Then, for $t \ge 1$, we have

$$\sum_{i=1}^{k-t} \binom{t+i-1}{t-1} s(k-1,t+i-1) = (k-1)s(k-2,t-1). \tag{24}$$

This we sum with (23). We find

$$\sum_{i=1}^{k-t} \binom{t+i}{t} s(k-1,t+i-1) =$$

$$(k-1)s(k-2,t-1) + (k-1)s(k-2,t) + s(k-1,t),\ t \ge 1. \tag{25}$$

Recall that ([1])

$$s(n,t) = s(n-1,t-1) - (n-1)s(n-1,t). \tag{26}$$

For $k \ne 1$, put here $n = k-1$ and multiply by $k-1$. We have

$$(k-1)s(k-1,t) =$$

$$(k-1)s(k-2,t-1) - (k-1)(k-2)s(k-2,t) =$$

$$(k-1)s(k-2,t-1) - ((k-1)^2 - (k-1))s(k-2,t),$$

whence

$$(k-1)^2 s(k-2,t) = (k-1)s(k-2,t-1)-$$

$$(k-1)s(k-1,t) + (k-1)s(k-2,t). \tag{27}$$

Taking into account the inductive supposition (21), from (27) we find

$$(k-1)\sum_{i=1}^{k-1-t}\binom{t+i}{t}s(k-1,t+i) =$$

$$(k-1)s(k-2,t-1) - (k-1)s(k-1,t) + (k-1)s(k-2,t). \tag{28}$$

Note that, since $s(k-1,k) = 0$, then in (28) we can consider the summing up to $i = k-t$. Subtracting (28) from (25), we have

$$\sum_{i=1}^{k-t}\binom{t+i}{t}(s(k-1,t+i-1) - (k-1)s(k-1,t+i)) = ks(k-1,t).$$

Since

$$s(k-1,t+i-1) - (k-1)s(k-1,t+i) = s(k,t+i),$$

then we find

$$\sum_{i=1}^{k-t}\binom{t+i}{t}s(k,t+i) = ks(k-1,t)$$

which, comparing with (21), means the step of induction. □

**Lemma 5.** *We have*

$$\sum_{i=1}^{k}(-1)^i(\binom{(n-1)i}{k} - \binom{ni}{k})\binom{k+1}{i+1} =$$

$$\sum_{i=1}^{k-1}(-1)^i\binom{ni}{k-1}\binom{k}{i+1}. \tag{29}$$

*Proof.* We prove (29) in the form

$$\sum_{i=1}^{k}(-1)^i\binom{(n-1)i}{k}\binom{k+1}{i+1} =$$

$$\sum_{i=1}^{k}(-1)^i\binom{ni}{k}\binom{k+1}{i+1} + \sum_{i=1}^{k-1}(-1)^i\binom{ni}{k-1}\binom{k}{i+1}. \tag{30}$$

According to (17) (which not depends on the validity of (13)), the coefficient of $n^t$ of right hand side of (30) equals $\frac{(-1)^t}{k!}s(k,t) + \frac{(-1)^t}{(k-1)!}s(k-1,t)$. Thus, by (30), we should prove that

$$Coef_{n^t}(\sum_{i=1}^{k}(-1)^i\binom{(n-1)i}{k}\binom{k+1}{i+1}) = \frac{(-1)^t}{k!}(s(k,t) + ks(k-1,t),$$

or

$$Coef_{n^t}(\sum_{i=1}^{k}(-1)^i\binom{k+1}{i+1})\sum_{r=0}^{k}s(k,r)((n-1)i)^r =$$
$$\sum_{i=1}^{k}(-1)^i\binom{k+1}{i+1})\sum_{r=t}^{k}s(k,r)i^r(-1)^{r-t}\binom{r}{t} =$$
$$(-1)^t(s(k,t)+ks(k-1,t),$$

or, changing the order of summing, equivalently we should prove that

$$\sum_{r=t}^{k}(-1)^r\binom{r}{t}s(k,r)\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}i^r = s(k,t)+ks(k-1,t) \tag{31}$$

(we can sum over $i \geq 0$, since $r \geq t \geq 1$). Note that the interior sum of (31) is

$$\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}i^r = \sum_{j=1}^{k+1}(-1)^{j-1}\binom{k+1}{j}(j-1)^r =$$
$$\sum_{j=0}^{k+1}(-1)^{j-1}\binom{k+1}{j}(j-1)^r + (-1)^r.$$

However, since $r \leq k$, then

$$\Delta^{k+1}[(j-1)^r] = \sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}(j-1)^r = 0$$

and thus

$$\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}i^r = (-1)^r.$$

Now the left hand side of (31) is $\sum_{r=t}^{k}\binom{r}{t}s(kr)$ and, by Lemma 4, is $s(k,t)+ks(k-1,t)$. $\square$

## 5. Completion of proof of Theorem 3

In Section 2 we proved that (5) is equivalent to (13). Therefore, our aim is to prove (13). We use induction over $k$. Note that (13), evidently, satisfies in case $k=1$ and every $n$. Suppose that (13) holds for $k := k-1$ and every $n$, i.e.,

$$\sum_{i=1}^{k-1}(-1)^i\binom{ni}{k-1}\binom{k}{i+1} = (-1)^{k-1}\binom{n+k-2}{k-1}. \tag{32}$$

By Lemma 5, the inductive supposition (32) is equivalent to the identity

$$\sum_{i=1}^{k}(-1)^i(\binom{(n-1)i}{k}-\binom{ni}{k})\binom{k+1}{i+1} =$$
$$(-1)^{k-1}\binom{n+k-2}{k-1}. \tag{33}$$

Putting $n := j$, and summing (33) over $j$ from $j = 1$ up to $j = n$, according to (6), we find

$$\sum_{i=1}^{k}(-1)^i\binom{ni}{k}\binom{k+1}{i+1} = (-1)^k\binom{n+k-1}{k}$$

which is realized the step of induction. $\square$

Simultaneously, in view of the proved in Section 3 equivalence of (13) and (18), we proved the identity (18).

## 6. Remarks on the newness of identities (13), (18) and (19)

Formally, the identities (13), (18) and (19) (and, consequently, (5)) appear to be new, since they are absent in so fundamental sources as [1],[4],[7]. However, there is a deeper reason. The newness of (13) (and together with it (18) and (19)) is explained by the fact that there are no known identities involving $\binom{in}{k}$ with the summing index $i$. Indeed, the only known generator of similar sums is Rothe-Hagen coefficient $A_k(x,n)$ [4]-[5]. It is defined alternatively by the following formulas:

$$A_k(x,n) = \frac{x}{x+kn}\binom{x+kn}{k}, \tag{34}$$

$$A_k(x,n) = \sum_{i=0}^{k-1}(-1)^{i+k+1}\binom{k}{i}\binom{x+in}{k}\frac{x}{x+in},\ k \geq 1. \tag{35}$$

The comparison of these formulas leads to the identity of the form

$$\sum_{i=1}^{k-1}(-1)^{i+k+1}\binom{k}{i}\binom{x+in}{k}\frac{x}{x+in} =$$

$$\frac{x}{x+kn}\binom{x+kn}{k} + (-1)^k\binom{x}{k}. \tag{36}$$

Unfortunately, the attempt to eliminate from $x$ in $\binom{x+in}{k}$, putting $x = 0$, lead to the trivial identity $0 = 0$. Consider another attempt. For $k > x \geq 1$, we have

$$\sum_{i=1}^{k-1}(-1)^{i+k+1}\binom{k}{i}\binom{x+in}{k}\frac{1}{x+in} = \frac{1}{x+kn}\binom{x+kn}{k},$$

or

$$\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}\binom{x+in}{k}\frac{1}{x+in} = 0,\ x \geq 1. \tag{37}$$

In the "*singular*" case $x = 0$, we obtain the required factor of the form $\binom{ni}{k}$ and found (quite independently on (37)) a nice identity

$$\sum_{i=1}^{k} \frac{(-1)^{i-1}}{i} \binom{in}{k} \binom{k}{i} = \frac{(-1)^{k-1} n}{k} \tag{38}$$

which, most likely, is also new, but different from (13). Indeed, denote the left hand side of (38) by $a_n(k)$. Using (14),we have

$$a_n(k) = \frac{1}{n!} \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} \binom{n}{i} (ik)(ik-1) \cdot \ldots \cdot (ik-n+1) =$$

$$\frac{1}{n!} \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} \binom{n}{i} \sum_{t=0}^{n} s(n,t)(ik)^t.$$

Thus, since $s(n,0)=0$, then

$$Coef_{k^t}(a_n(k)) = \begin{cases} 0, \ \ if \ \ t=0, \\ \frac{s(n,\,t)}{n!} \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} i^{t-1}, \ \ if \ \ t \geq 1. \end{cases} \tag{39}$$

Further, since

$$s(n,1) = (-1)^{n-1}(n-1)!, \ \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} = 1,$$

then

$$Coef_k(a_n(k)) = \frac{(-1)^{n-1}}{n}. \tag{40}$$

It is left to show that, for $t \geq 2$, we have

$$s(n,\ t) \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} i^{t-1} = 0. \tag{41}$$

Indeed, if $2 \leq t \leq n$, then we have

$$\sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} i^{t-1} = (-1)^{n-1} \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (k-i)^{t-1}.$$

The latter is the $n$-th difference $\Delta^n[k^{t-1}]$ which, for $t \leq n$, equals 0. If $t > n$, then $s(n,\ t) = 0$, and (41) follows. $\square$

## 7. Gessel's short proof of (13)

Gessel [3] proposed a short proof of the identity (13).

Let $P(x)$ be a polynomial of degree $k$. Then, for the $(k+1)$-th difference of $P(x)$, we have

$$\Delta^{k+1}[P(x)] = \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P(x+j) = 0.$$

In particular, for $x = 0$,

$$\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}P(j)=0.$$

Put here

$$P(j)=P_{n,\,k}(j)=\binom{n(j-1)}{k}$$

which is a polynomial in $j$ of degree $k$. We have

$$\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}\binom{n(j-1)}{k}=0.$$

Putting here $j-1=i$, we find

$$\sum_{i=-1}^{k}(-1)^i\binom{k+1}{i+1}\binom{ni}{k}=0,$$

or, the same, for $k\geq 1$, we have

$$\sum_{i=1}^{k}(-1)^i\binom{k+1}{i+1}\binom{ni}{k}=\binom{-n}{k}=$$

$$\frac{(-n)(-n-1)\cdot\ldots\cdot(-n-(k-1))}{k!}=$$

$$(-1)^k\frac{(n+k-1)(n+k-2)\cdot\ldots\cdot n}{k!}=(-1)^k\binom{n+k-1}{k}. \quad \square$$

It is interesting to note that, if the author was successful to find such an elegant and simple proof, then, most likely, the identities (18), (19) and (38) were not discovered.

## 8. Dual case of identity (5)

Note that, together with Example1, we have the following identity

$$\binom{n}{1}^3-2\binom{n}{1}\binom{n+1}{2}+\binom{n+2}{3}=\binom{n}{3}.$$

In general, together with (5), we prove the following dual identity.

**Theorem 6.**

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+\ldots+k_r=k,\;k_i\geq 1}\;\prod_{i=1}^{r}\binom{n+k_i-1}{k_i}=\binom{n}{k}. \tag{42}$$

*Proof.* Again we calculate the interior sum in (42) in a combinatorial way with firstly consideration also zero parts in the compositions of $k$. In this case we have the sum

$$\Sigma_3 = \sum_{k_1+k_2+...+k_r=k,\ k_i\geq 0}\ \prod_{i=1}^{r}\binom{n+k_i-1}{k_i}. \tag{43}$$

To calculate this sum, suppose that we have $rn$ white points and mark $k$ from them, *but now every point could be marked several times.* According to well known formula for the number of the combination with repetitions (cf. [8], p.10), this we can do in $\binom{rn+k-1}{k}$ ways. On the other hand, we can mark (with repetitions) $k_1$ from $n$ points, $k_2$ from another $n$ points, etc. This leads us to the equality

$$\Sigma_3 = \binom{rn+k-1}{k}. \tag{44}$$

To calculate the required interior sum in (42)

$$\Sigma_4 = \sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\ \prod_{i=1}^{r}\binom{n+k_i-1}{k_i}, \tag{45}$$

we should remove zero parts in $\Sigma_3$ (44), using "include-exclude" formula. We find

$$\Sigma_4 = \binom{rn+k-1}{k} - \binom{r}{1}\binom{(r-1)n+k-1}{k} +$$

$$\binom{r}{2}\binom{(r-2)n+k-1}{k} - ... + (-1)^{r-1}\binom{r}{r-1}\binom{n+k-1}{k}. \tag{46}$$

Now in an analogous way, as in Section 2, we find that the identity (42) is equivalent to the identity dual to (13):

$$\sum_{i=1}^{k}(-1)^i\binom{ni+k-1}{k}\binom{k+1}{i+1} = (-1)^k\binom{n}{k}. \tag{47}$$

The latter identity is easily proved as (13) in Section 7. □

## 9. An algebraic approach

L. Tevlin [10] outlined the contours of quite another proof of Theorem 3 in frameworks of the good advanced theory of symmetric functions. Recall (cf.[6], [8]) that, for each integer $k \geq 0$,

$i$) the $k$-th elementary symmetric function $e_k$ is the sum of all products of $k$ distinct variables $x_i$, so that $e_0 = 1$ and, for $k \geq 1$,

$$e_k = \sum_{i_1<i_2<...<i_k} x_{i_1}x_{i_2}...x_{i_k}; \tag{48}$$

$ii$) the $k$-th complete symmetric function $h_k$ is defined as $h_0 = 1$ and, for $k \geq 1$,

$$h_k = \sum_{i_1\leq i_2\leq...\leq i_k} x_{i_1}x_{i_2}...x_{i_k}. \tag{49}$$

In particular, $h_1 = e_1$. It is convenient to define $e_k = h_k = 0$ for $k < 0$;
*iii*) it is well known that

$$h_k = \begin{vmatrix} e_1 & e_2 & e_3 & \dots & e_{k-2} & e_{k-1} & e_k \\ 1 & e_1 & e_2 & \dots & e_{k-3} & e_{k-2} & e_{k-1} \\ 0 & 1 & e_1 & \dots & e_{k-4} & e_{k-3} & e_{k-2} \\ 0 & 0 & 1 & \dots & e_{k-5} & e_{k-4} & e_{k-3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1 & e_1 & e_2 \\ 0 & 0 & 0 & \dots & 0 & 1 & e_1 \end{vmatrix} \tag{50}$$

and

$$e_k = \begin{vmatrix} h_1 & h_2 & h_3 & \dots & h_{k-2} & h_{k-1} & h_k \\ 1 & h_1 & h_2 & \dots & h_{k-3} & h_{k-2} & h_{k-1} \\ 0 & 1 & h_1 & \dots & h_{k-4} & h_{k-3} & h_{k-2} \\ 0 & 0 & 1 & \dots & h_{k-5} & h_{k-4} & h_{k-3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1 & h_1 & h_2 \\ 0 & 0 & 0 & \dots & 0 & 1 & h_1 \end{vmatrix}. \tag{51}$$

Diagonals of these determinants has very simple cycle structure that allows to give an explicit formulas for them.

**Lemma 7.** *The following formulas hold*

$$h_k = \sum_{r=1}^{k} (-1)^{k-r} \sum_{k_1+k_2+\dots+k_r=k,\ k_i \geq 1} \prod_{i=1}^{r} e_{k_i}; \tag{52}$$

$$e_k = \sum_{r=1}^{k} (-1)^{k-r} \sum_{k_1+k_2+\dots+k_r=k,\ k_i \geq 1} \prod_{i=1}^{r} h_{k_i}. \tag{53}$$

*Proof.* Consider products of nonzero elements of cycles of the Toeplitz matrix (50). It is easy to see that for every cycle of length 1 this product is $e_1$, for every cycle of length 2 this product is $e_2$, ..., for every cycle of length $i$ this product is $e_i$. Therefore, a diagonal with cycles of length $k_1, k_2, ..., k_r$, such that $k_1 + k_2 + ... + k_r = k$, has the product of its elements $\prod_{i=1}^{r} e_{k_i}$ and in the determinant this product appears with sign $(-1)^{k-r}$. Hence, (52) follows. Dually we have also (53). $\square$

However, by Ex.1, p.26 in [6], it follows that, if $e_k = \binom{n}{k}$, then $h_k = \binom{n+k-1}{k}$. Thus, in view of Lemma 7, we obtain new proofs of identities (5) and (42).

## 10. Other identities generated by compositions of integers

In [6] we find also other pairs $\{e_k, h_k\}$ given by explicit formulas. So we obtain other interesting identities generated by compositions of integers. We restrict ourself by the following five pairs of identities.

1) Pair $e_k=\frac{a(a-k)^{k-1}}{k!},\ h_k=\frac{a(a+k)^{k-1}}{k!},\ k\geq 1$, leads to identities:

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}\frac{a(a-k_i)^{k_i-1}}{k_i!}=\frac{a(a+k)^{k-1}}{k!}; \tag{54}$$

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}\frac{a(a+k_i)^{k_i-1}}{k_i!}=\frac{a(a-k)^{k-1}}{k!}; \tag{55}$$

2) Pair $e_k=\frac{(-1)^k a^k B_k}{k!},\ h_k=\frac{a^k}{(k+1)!},\ k\geq 1$, where $B_k$ is the $k$-th Bernoulli number, leads to identities:

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}\frac{(-1)^{k_i}a^{k_i}B_{k_i}}{k_i!}=\frac{a^k}{(k+1)!}; \tag{56}$$

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}\frac{a^{k_i}}{(k_i+1)!}=\frac{(-1)^k a^k B_k}{k!}; \tag{57}$$

3) Pair $e_k=q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix},\ h_k=\begin{bmatrix}n+k-1\\k\end{bmatrix}$, where $\begin{bmatrix}n\\k\end{bmatrix}$ denotes the "$q$-binomial coefficient" or Gaussian polynomial

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q)(1-q^2)...(1-q^k)},$$

leads to identities:

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}q^{\frac{k_i(k_i-1)}{2}}\begin{bmatrix}n\\k_i\end{bmatrix}=\begin{bmatrix}n+k-1\\k\end{bmatrix}; \tag{58}$$

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}\begin{bmatrix}n+k_i-1\\k_i\end{bmatrix}=q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}; \tag{59}$$

4) Pair $e_k=q^{\frac{k(k-1)}{2}}/\varphi_k(q),\ h_k=1/\varphi_k(q)$, where

$$\varphi_k(q)=(1-q)(1-q^2)...(1-q^k),$$

leads to identities:

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}(q^{\frac{k_i(k_i-1)}{2}}/\varphi_{k_i}(q))=1/\varphi_k(q); \tag{60}$$

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+...+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}1/\varphi_{k_i}(q)=q^{\frac{k(k-1)}{2}}/\varphi_k(q); \tag{61}$$

5) Pair $e_k=\prod_{i=1}^{k}\frac{a-bq^{i-1}}{1-q^i},\ h_k=\prod_{i=1}^{k}\frac{aq^{i-1}-b}{1-q^i}$, leads to identities:

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+\ldots+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}\prod_{j=1}^{k_i}\frac{a-bq^{j-1}}{1-q^j}=\prod_{i=1}^{k}\frac{aq^{i-1}-b}{1-q^i}; \tag{62}$$

$$\sum_{r=1}^{k}(-1)^{k-r}\sum_{k_1+k_2+\ldots+k_r=k,\ k_i\geq 1}\prod_{i=1}^{r}\prod_{j=1}^{k_i}\frac{aq^{j-1}-b}{1-q^j}=\prod_{i=1}^{k}\frac{a-bq^{i-1}}{1-q^i}. \tag{63}$$

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